Exercise 25

Sketch the graph of a function g that is continuous on its domain $(-5, 5)$ and where $g(0) = 1$, $g'(0) = 1$, $g'(-2) = 0$, $\lim_{x\to-5^+} g(x) = \infty$, and $\lim_{x\to5^-} g(x) = 3$.

Solution

Use the following trial function with four constants to satisfy the five conditions.

$$g(x) = \frac{1}{x+5} + Ax^3 + Bx^2 + Cx + D$$

The first term, $1/(x + 5)$, automatically satisfies $\lim_{x\to-5^+} g(x) = \infty$, so only four constants are necessary. Take the derivative of $g(x)$.

$$g'(x) = -\frac{1}{(x+5)^2} + 3Ax^2 + 2Bx + C$$

Apply the remaining four conditions to get a system of equations for the four unknowns.

$$\begin{cases} g(0) = \dfrac{1}{(0)+5} + A(0)^3 + B(0)^2 + C(0) + D = 1 \\ \lim\limits_{x\to5^-} g(x) = \dfrac{1}{(5)+5} + A(5)^3 + B(5)^2 + C(5) + D = 3 \\ g'(0) = -\dfrac{1}{(0+5)^2} + 3A(0)^2 + 2B(0) + C = 1 \\ g'(-2) = -\dfrac{1}{(-2+5)^2} + 3A(-2)^2 + 2B(-2) + C = 0 \end{cases}$$

Solving this system yields

$$\begin{cases} A = -\dfrac{1603}{36\,000} \\ B = \dfrac{3551}{36\,000} \\ C = \dfrac{26}{25} \\ D = \dfrac{4}{5} \end{cases}.$$

Now that the constants are known, the function is known and can be plotted versus x.

$$g(x) = \frac{1}{x+5} - \frac{1603}{36\,000}x^3 + \frac{3551}{36\,000}x^2 + \frac{26}{25}x + \frac{4}{5}$$